

CONFORMAL BLOCKS DIVISORS ON $\overline{M}_{0,n}$ FROM \mathfrak{sl}_2

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ABSTRACT. We study a family of semiample divisors on $\overline{M}_{0,n}$ defined using conformal blocks and analyze their associated morphisms.

1. INTRODUCTION

The coarse moduli space $\overline{M}_{g,n}$ is a projective variety whose points correspond to isomorphism classes of Deligne-Mumford stable n -pointed curves of genus g . Our collective intuition for what moduli spaces should look like is based on the spaces $\overline{M}_{g,n}$, and these varieties have been studied from many perspectives. One of the most important problems is to describe all morphisms admitted by $\overline{M}_{g,n}$. Maps to projective varieties are given by the section rings of divisors. A divisor D is *semiample* if some multiple of D is base point free; the map associated to a semiample divisor is therefore a morphism. Semiample divisors lie in the nef cone, the set of divisors that nonnegatively intersect all curves on the variety.

For the special case when $g = 0$, $\overline{M}_{0,n}$ is conjectured by Hu and Keel to be a “Mori Dream Space,” which would imply that every nef divisor on $\overline{M}_{0,n}$ is semiample and that the cone of nef divisors is finitely generated [19]. The major support for the conjecture comes from our understanding of the space for $n = 4, 5, 6$. For example, we know that $\overline{M}_{0,6}$ is a Mori dream space from two perspectives: Castravet gives an explicit constructive proof in [7], and the sweeping theory of [5] shows that log Fano varieties are Mori dream spaces. Unfortunately since $\overline{M}_{0,n}$ is not log Fano for $n \geq 7$, one cannot directly apply the results of [5], and extending Castravet’s approach to larger values of n seems daunting. Attempts to prove that the conjectural descriptions of the nef and effective cones are correct seem mired down in the fiercely combinatorial nature of these moduli spaces.

Very recently Fakhruddin, in [11], showed that there is a potentially infinite supply of semiample divisors on $\overline{M}_{0,n}$ that arise as first Chern classes of vector bundles which come from the theory of conformal blocks. First defined by Tsuchiya, Ueno, and Yamada, these bundles are specified by a simple Lie algebra \mathfrak{g} , a positive integer ℓ called the *level*, and an appropriately chosen n -tuple of dominant weights $\vec{\lambda}$. The ranks of these vector bundles are computed by the famous Verlinde formula. Fakhruddin has given recursive formulas for the classes of these divisors and for their intersection with certain combinatorially defined curves in called F-curves. In this work we study some of the simplest examples of these divisors, determine where they lie in the cone of nef divisors, and describe the morphisms associated to them.

We consider a family of divisors on $\overline{M}_{0,n}$ determined by the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. In this case, the weights $\vec{\lambda}$ can be identified with an n -tuple of nonnegative integers less than or equal to the level ℓ . We take all the weights to be equal to one, and examine the family one obtains by varying ℓ . As we now explain, this gives a collection of numerical equivalence classes of semiample divisors $\{\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} : 1 \leq \ell \leq g\}$ on $\overline{M}_{0,n}$ with $n = 2(g+1)$ even. Indeed, if n is odd, then by Lemma 3.6

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below, one has that $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ is trivial. Also, Fakhruddin has shown that if $2\ell \geq \sum_i \lambda_i$, then $\mathbb{D}_{\ell,\vec{\lambda}}^{\mathfrak{sl}_2}$ is trivial. In particular, if $\ell \geq g+1$ then the divisor $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ is trivial.

One can work with these divisors as completely combinatorial objects. We give a formula for intersecting these divisors with a basis of 1-cycles on the space and give a closed form expression for their classes (Theorem 4.2, Propositions 5.1 and 5.2). Using this information, we study the position of these divisors in the nef cone and study the morphisms they define.

Although these divisors arise from a simple choice of weights, they have many interesting geometric properties. The symmetric group S_n acts on $\overline{M}_{0,n}$ by permuting the marked points and the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ are S_n invariant. In particular, by [21] (and [15]), each of these divisors is big, and hence the morphism it defines is birational. Divisors that give embeddings lie on the interior of the nef cone. As we show, each member $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ of our family lies on the boundary of $\text{Nef}(\overline{M}_{0,n}/S_n)$, and defines a birational contraction of $\overline{M}_{0,n}/S_n$. In particular, four of the divisors generate extremal rays of the symmetric nef cone (Theorem 6.1).

These divisors define geometrically meaningful maps. For example:

- $\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2}$ defines a morphism from $\overline{M}_{0,n}$ to the Satake compactification of the moduli space of abelian varieties of dimension g (Theorem 7.2).
- $\mathbb{D}_{g-1,(1,\dots,1)}^{\mathfrak{sl}_2}$ defines the morphism from $\overline{M}_{0,n}$ to a natural (generalized) flip of the GIT quotient $(\mathbb{P}^1)^n // \text{SL}_2$ (Theorem 7.7).
- $\mathbb{D}_{g,(1,\dots,1)}^{\mathfrak{sl}_2}$ defines the morphism from $\overline{M}_{0,n}$ to the GIT quotient $(\mathbb{P}^1)^n // \text{SL}_2$ (Theorem 7.5).

In addition, we recall that Fakhruddin shows that the morphism given by any $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ for $\ell \leq g-1$ factors through Hassett's reduction morphisms ([18]) $\rho_A : \overline{M}_{0,n} \rightarrow \overline{M}_{0,A}$ for $A = (\frac{1}{\ell+1}, \dots, \frac{1}{\ell+1})$ [11, Prop. 4.7].

Using each of the four divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$, $\ell = 1, 2, g-1, g$, we obtain a family of nef divisors on $\overline{M}_{2(g+1)}$, the moduli space of stable curves of genus $2(g+1)$; we give these classes explicitly in Proposition 7.10.

The Ray Theorem of Keel and McKernan [21, Thm. 1.2] and its extension by Farkas and Gibney [12, Thm. 4] enable one to give a combinatorial proof that divisors in what we call the log canonical part of the cone are nef. If a divisor D is of this form, and if D intersects every F-curve nonnegatively, then D is nef. As we show in Proposition 6.3, there are divisors in our family that do not lie in this part of the cone. In particular, we know no combinatorial proof that they are semiample.

There are two other families of conformal blocks divisors that have been studied extensively. In [11], Fakhruddin considers the set of divisors given by \mathfrak{sl}_2 , level $\ell = 1$ and varying weights $\vec{\lambda}$. He shows these divisors form a basis for $\text{Pic}(\overline{M}_{0,n})$ and that all of the morphisms factor through the reduction maps from $\overline{M}_{0,n}$ to Hassett's moduli spaces of stable weighted pointed rational curves. In [3], the set of divisors $\mathbb{D}_{1,j}^{\mathfrak{sl}_n}$ given by \mathfrak{sl}_n , level $\ell = 1$ and symmetric sets of fundamental dominant weights $\vec{\lambda} = \{\omega_j, \dots, \omega_j\}$ is considered. The authors show that each of these divisors is extremal in the symmetric nef cone and so defines a morphism to a variety of Picard number one. In [14], Giansiracusa describes the images of the morphisms given by the $\mathbb{D}_{1,j}^{\mathfrak{sl}_n}$.

Outline of paper: In Section 2 we give definitions and references for divisors and curves on $\overline{M}_{g,n}$. In Theorem 2.1 we define three families of independent curves on $\overline{M}_{0,n}$ that we use to prove that the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ are extremal in the symmetric nef cone. In Section 3.1 we provide references and brief definitions for general conformal blocks. Some of our main technical lemmas about \mathfrak{sl}_2 conformal blocks are proved in Section 3.2, where we recall the Verlinde formula, as well as the factorization and fusion rules. In Section 4 we give the intersection formulas for the S_n -invariant

divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ with a basis for the 1-cycles on $\overline{M}_{0,n}/S_n$. As an application we show that the set of nontrivial divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ forms a basis of $\text{Pic}(\overline{M}_{0,n}/S_n)$. In Section 5 we prove Proposition 5.1, which gives a general formula for the classes of the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$. This expression depends on ranks of certain \mathfrak{sl}_2 conformal blocks bundles which we give in Proposition 5.2. In Section 5.2 we give simplified expressions for the four elements of the family that generate extremal rays of the symmetric nef cone and we give simplified versions of the classes of some of the divisors that lie on higher dimensional extremal faces. In Section 6 we show that the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ are extremal in the symmetric nef cone and that they reside in the part of the cone that was not previously well understood. Finally, in Section 7 we study morphisms defined by the extremal divisors.

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2. DIVISORS AND CURVES ON THE MODULI STACK

The stack $\overline{\mathcal{M}}_{g,n}$, which parametrizes flat families of stable curves, reflects the geometry of the variety $\overline{M}_{g,n}$; throughout this paper, we will sometimes work on the space and sometimes on the stack. In genus 0, $\overline{M}_{0,n}$ is actually a fine moduli space, and so these two points of view are equivalent in this particular case.

The conformal blocks divisors are defined on the moduli stack $\overline{\mathcal{M}}_{g,n}$ as the determinants of vector bundles of conformal blocks. We will often express them in terms of the tautological and boundary classes. In Section 2.1, we record the basic definitions of these classes and refer the reader to [4] for details. In Section 2.2 we define three collections of curves on $\overline{M}_{0,n}$ that we prove are independent. These curves are used to analyze the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$.

2.1. Divisor classes. We write λ for the first Chern class of the Hodge bundle, as is standard in the literature. For $1 \leq i \leq n$, we denote by σ_i the n sections of the universal family $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$. Then for ω_π the rank 1 relative dualizing sheaf, one has the tautological classes $\psi_i = c_1(\sigma_i^*(\omega_\pi))$. We refer to the sum $\Psi = \sum_{i=1}^n \psi_i$ as the total Ψ class. The divisor $\kappa = \kappa_1 = \pi_*(\omega_\pi^2)$ is ample.

We write δ_0 for the class of the boundary component Δ_0 , the divisor whose general element has a single nonseparating node. For $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$, and $J \subset \{1, \dots, n\}$, let $\delta_{i,J}$ be the class of the boundary divisor $\Delta_{i,J}$. The general element of $\Delta_{i,J}$ has a single separating node that breaks the curve into two components, one of which is a curve of genus i and has $|J| + 1$ marked points consisting of an attaching point together with points labeled by the set J . If $n = 0$, it is customary to write δ_i instead of $\delta_{i,\emptyset}$ and if $g = 0$, it is customary to write δ_J rather than $\delta_{0,J}$. By Δ we mean the sum of all the boundary divisors.

As we often work with S_n -invariant divisor classes on $\overline{M}_{0,n}$, we often find it convenient to write

$$B_j = \sum_{\substack{J \subset \{1, \dots, n\} \\ |J|=j}} \delta_J.$$

In [21, Thm. 1.3], the B_j were shown to generate the extremal rays of the cone of effective divisors of the quotient $\overline{M}_{0,n}/S_n$. These form a basis for $\text{Pic}(\overline{M}_{0,n})^{S_n}$.

Finally, by [25, Theorem 1] the class of the canonical divisor in this basis is

$$K_{\overline{M}_{0,n}} = \Psi - 2\Delta = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{i(n-i)}{(n-1)} - 2 \right) B_i.$$

2.2. Three collections of independent curves. An F-curve on $\overline{M}_{0,n}$ is any curve that is numerically equivalent to a 1-dimensional component of the boundary. Its class depends only on a partition of the marked points into four nonempty sets. As we consider the intersection of F-curves with symmetric divisor classes on $\overline{M}_{0,n}$, we do not need to know the partition itself, but only the size of the cells of the partition. Thus, a partition $a+b+c+d=n$ of the integer n into four positive integers determines an F-curve class, up to S_n symmetry. We denote such a curve by $F_{a,b,c,d}$, or even more briefly by $F_{a,b,c}$, with the fourth integer $d=n-(a+b+c)$ implicit. Details and a definition of F-curve on $\overline{M}_{g,n}$ for all g and all n are given in [16, Thm. 2.2, Fig. 2.3].

We next define three families of F-curves on $\overline{M}_{0,n}$ that we show are independent. These curves are used to demonstrate in Theorem 5.19 that the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\text{sl}_2}$ are extremal in the symmetric nef cone.

Theorem 2.1. *Let $n=2g+2$, or $n=2g+3$. Each of the following three sets consist of independent curves.*

- (1) $\mathcal{C}_1 = \{F_{1,1,i} : 1 \leq i \leq g\};$
- (2) $\mathcal{C}_2 = \{F_{2,2,i} : 1 \leq i \leq g-1\};$ and
- (3) $\mathcal{C}_3 = \{F_{3,3,2i+1} : 0 \leq i \leq k-2\} \cup \{F_{1,1,2i+1} : 0 \leq i \leq k-1\},$ and $g=2k$ or $g=2k-1$.

This leads to the following well-known corollary:

Corollary 2.2. *Let $n=2(g+1)$ or $n=2(g+1)+1$ and \mathcal{C}_1 as in Theorem 2.1. Then \mathcal{C}_1 is a basis for $N_1(\overline{M}_{0,n}/S_n, \mathbb{Q})$.*

Proof of Corollary 2.2. The Néron-Severi space of $\text{NS}(\overline{M}_{0,n}/S_n)_{\mathbb{Q}}$ is g -dimensional. By Theorem 2.1, \mathcal{C}_1 consists of g independent curves. \square

Proof of Theorem 2.1. For a proof that the first family \mathcal{C}_1 is independent, see [3, Prop. 4.1].

To show that \mathcal{C}_2 is a collection of independent curves, suppose that $R = \sum_{i=1}^g b_i F_{2,2,i}$ is equivalent to zero and put $B = \sum_{i=1}^g b_i$. By intersecting R with the boundary classes B_i , one gets the following equations

- (1) $0 = R \cdot B_2 = -2B - b_2;$
- (2) $0 = R \cdot B_3 = 2b_1 - b_3;$
- (3) $0 = R \cdot B_4 = B + 2b_2 - b_4;$
- (4) for $5 \leq i \leq g-3$, and if $n=2g+3$ and $i \in \{g-2, g-1\}$, $0 = R \cdot B_i = 2b_{i-2} - b_i - b_{i-4};$
- (5) for $n=2g+2$,
 - (a) $0 = R \cdot B_{g-2} = 2b_{g-4} - b_g - b_{g-2} - b_{g-6};$
 - (b) $0 = R \cdot B_{g-1} = 2b_{g-3} - 2b_{g-1} - b_{g-5};$
 - (c) $0 = R \cdot B_g = b_{g-2} - b_g - b_{g-4};$
 - (d) $0 = R \cdot B_{g+1} = 2b_{g-1} - b_{g-3};$
- (6) for $n=2g+3$,
 - (a) $0 = R \cdot B_g = 2b_{g-2} - b_g - b_{g-1} - b_{g-4};$
 - (b) $0 = R \cdot B_{g+1} = 2b_{g-1} - b_{g-3} - b_{g-2}.$

By adding all of these relations together one gets that $B = -b_1$. We now show that all of the coefficients are a multiple of b_1 . Iteration of equation (4) gives that for $5 \leq i \leq g-3$, and if $n=2g+3$ and $i \in \{g-2, g-1\}$, then for $m \leq \lfloor \frac{i+1}{2} \rfloor$, one has $b_i = mb_{i-2m+2} - (m-1)b_{i-2m}$. For i odd, taking $m = \frac{i-1}{2}$, and using equation (2) gives that $b_i = \frac{i+1}{2}b_1$. For i even, taking $m = \frac{i-2}{2}$, and using equations (3) and then (1) gives that

$$b_i = \frac{i-2}{2}b_4 - \frac{i-6}{2}b_2 = \frac{i-2}{2}(B + 2b_2) - \frac{i-6}{2}b_2 = \frac{i-2}{2}B + \frac{i+2}{2}b_2 = -\frac{i+6}{2}B = \frac{i+6}{2}b_1.$$

Using the remaining equations it is very easy to get expressions for the rest of the coefficients in terms of b_1 . To finish the proof, we intersect R with the divisor $\psi = \sum_{i=1}^n \psi_i$. Since R is assumed

trivial, $R \cdot \psi = 0$. On the other hand, one can easily check that ψ intersects R in degree b_1 , proving that b_1 must be zero. We conclude that all the coefficients are zero, and the curves are independent.

To show that \mathcal{C}_3 consists of $2k - 1$ independent curves, suppose that

$$R = \sum_{i=0}^{k-1} a_i F_{1,1,2i+1} + \sum_{i=0}^{k-2} b_i F_{3,3,2i+1}$$

is equivalent to zero. Put $A = \sum_{i=0}^{k-1} a_i$ and $B = \sum_{i=0}^{k-2} b_i$. By intersecting R with the boundary classes B_i , one gets the following equations

- (1) $0 = R \cdot B_2 = A + 2a_0$;
- (2) $0 = R \cdot B_3 = -2B - a_0 - (a_1 + b_1)$;
- (3) $0 = R \cdot B_{2j} = 2(a_{j-1} + b_{j-2})$, for $2 \leq j \leq k$ and $j \neq 3$.
- (4) $0 = R \cdot B_{2j+1} = -(a_{j-1} + b_{j-3} + a_j + b_j)$, for $2 \leq j \leq k$;
- (5) $0 = R \cdot B_6 = B + 2(a_2 + b_1)$.

By adding all of these relations together one gets that $A + a_0 + b_0 - B = 0$. Intersecting R with $\psi = \sum_{i=1}^n \psi_i$ gives that $0 = A + a_0 + b_0$. Putting these two facts together gives that $B = 0$ and using the first equation, which says that $A = -2a_0$, we can write $b_0 = a_0$. We will next show $b_j = 0$ for $1 \leq j \leq k - 1$. We work by induction on j , with base case $j = 1$. To begin with, we can see that $a_0 = -a_1$. Indeed, we have seen that $b_0 = a_0$ and equation (3) says that $b_0 = -a_1$. Now equation (2) says that $b_1 = -a_0 - a_1$, and so the base case holds. So we fix $j \in \{2, \dots, k - 1\}$ and assume that $b_\ell = 0$ if $\ell < j$. Now by equations (4), and then (3), one has that $b_j = -(a_{j-1} + b_{j-3} + a_j) = b_{j-2} + b_{j-3} + b_{j-1}$, which is zero by our induction hypothesis. Finally, since $b_j = 0$, we may use equation (3) to see that the a_j are zero as well. \square

3. CONFORMAL BLOCKS

The facts we use about conformal blocks divisors can be found in [6], [24], [30], and [11]. In Section 3.1 we give an informal description of conformal blocks. In Section 3.2 we give specific results about \mathfrak{sl}_2 conformal blocks, including the factorization and fusion rules and other technical lemmas we use throughout the paper.

3.1. General definition. Let \mathfrak{g} be a simple Lie algebra, ℓ a positive integer called the *level*, and $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ an n -tuple of dominant integral weights for \mathfrak{g} of level ℓ . Conformal blocks divisors are determinants of vector bundles $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ defined on the stack $\overline{\mathcal{M}}_{g,n}$ which are determined by the data \mathfrak{g} , ℓ , and $\vec{\lambda}$. We write:

$$\mathbb{D}_{\ell, \vec{\lambda}}^{\mathfrak{g}} = c_1(\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})).$$

We often refer to the $\mathbb{D}_{\ell, \vec{\lambda}}^{\mathfrak{g}}$ as CB-divisors and the $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ as CB-bundles. The divisors $\mathbb{D}_{\ell, \vec{\lambda}}^{\mathfrak{g}}$ are also known in the literature as generalized theta divisors.

We give two descriptions of the fiber of the CB-bundle $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ over a smooth point $(C, \vec{p}) = (C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$. First, a geometric description [23, Eq. (1.2.2)]:

$$\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})|_{(C, \vec{p})} \cong H^0(\mathcal{M}_{\mathfrak{g}}^{\text{par}}(C, \vec{p}), \mathcal{L}).$$

Here $\mathcal{M}_{\mathfrak{g}}^{\text{par}}(C, \vec{p})$ is a moduli stack parametrizing quasi-parabolic \mathfrak{g} -bundles on C determined by $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$. The line bundle \mathcal{L} is determined by ℓ .

Second, we give an algebraic description, which comes from the construction of these CB-bundles using the representation theory of affine Lie algebras. To define them, let

$$\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}((z_i))) \oplus \mathbb{C}c$$

be the affine Lie algebra associated to \mathfrak{g} . As for finite-dimensional Lie algebras, to each weight λ_i there is an associated irreducible $\widehat{\mathfrak{g}}$ -module \mathcal{H}_{λ_i} . Let $\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \dots \otimes \mathcal{H}_{\lambda_n}$. For a smooth point

$(C; \{p_1, \dots, p_n\}) \in \mathcal{M}_{g,n}$, set $U = C - \{p_1, \dots, p_n\}$. Choose a local coordinate at each point p_i . This gives rise to a ring homomorphism $\mathcal{O}(U) \rightarrow \mathbb{C}((z))$ for each $i = 1, \dots, n$ mapping a function f to its Laurent series expansion f_{p_i} around p_i . A $\mathfrak{g} \otimes \mathcal{O}(U)$ action on $\mathcal{H}_{\vec{\lambda}}$ is defined by the formula

$$(X \otimes f) \cdot (v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \dots \otimes (X \otimes f_{p_i})v_i \otimes \dots \otimes v_n,$$

and the fiber of $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ over the point (C, p_1, \dots, p_n) is the vector space of coinvariants $\mathcal{H}_{\vec{\lambda}}/(\mathfrak{g} \otimes \mathcal{O}(U))\mathcal{H}_{\vec{\lambda}}$. These fibers are independent of the choice of local coordinate around p_i and vary nicely in families to form an algebraic vector bundle on $\mathcal{M}_{g,n}$. The construction can be extended to nodal curves, and yields an algebraic vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ on the stack $\overline{\mathcal{M}}_{g,n}$.

3.2. \mathfrak{sl}_2 CB-bundles. In this section we recall the Verlinde formula, as well as the factorization and fusion rules for \mathfrak{sl}_2 CB-bundles. Using factorization and the fusion rules, we prove Lemmas 3.6, 3.8, and 3.9, which are all results about ranks of particular \mathfrak{sl}_2 CB-bundles that considered in this work. We also prove Lemma 3.12, which gives a simple formula for the degrees of a certain CB bundles on $\mathbb{P}^1 \cong \overline{M}_{0,4}$.

The root system of $\mathfrak{g} = \mathfrak{sl}_2$ may be identified with \mathbb{Z} , and dominant integral weights λ_i of level ℓ are simply nonnegative integers $0 \leq \lambda_i \leq \ell$. Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a vector of dominant integral weights of level ℓ .

Throughout this section work only with \mathfrak{sl}_2 , and we work with an arbitrary (but fixed) level ℓ . Therefore we will abbreviate our notation a little.

Notation We write

$$(3.1) \quad r_{\vec{\lambda}} := \text{rank } \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\lambda}).$$

For $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$, the rank of the vector bundle $\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\lambda})$ is given by the Verlinde formula:

$$(3.2) \quad r_{\vec{\lambda}} = \left(\frac{\ell+2}{2}\right)^{g-1} \sum_{j=0}^{\ell} \frac{\prod_{i=1}^n \sin\left(\frac{(\lambda_i+1)(\lambda_j+1)\pi}{\ell+2}\right)}{\left(\sin\left(\frac{(\lambda_j+1)\pi}{\ell+2}\right)\right)^{2g+n-2}}.$$

Although this formula is quite elegant, it is often computationally more efficient to use the factorization rules. These may be stated for any simple Lie algebra \mathfrak{g} , but we will only work with $\mathfrak{g} = \mathfrak{sl}_2$.

Proposition 3.3 (Propagation for CB-bundles). *Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$, and suppose that $\lambda_n = 0$. Then $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda}) = \pi_n^* \mathbb{V}(\mathfrak{g}, \ell, \hat{\lambda})$, where $\hat{\lambda} = (\lambda_1, \dots, \lambda_{n-1})$ and $\pi_n : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n-1}$ is the map forgetting the n th marked point. In particular, $r_{\vec{\lambda}} = r_{\hat{\lambda}}$.*

Proposition 3.4 (Factorization for \mathfrak{sl}_2 CB-bundles). *Let $\vec{\mu} \cup \vec{\nu}$ be a partition of the vector $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ into two vectors each of length at least 2. Then*

$$r_{\vec{\lambda}} = \sum_{\alpha=0}^{\ell} r_{\vec{\mu} \cup \alpha} r_{\vec{\nu} \cup \alpha}.$$

(In general, statements of factorization for CB-bundles insert weights α and α^* , where $*$ denotes the involution on the root lattice defined by $\alpha^* = \sigma(\alpha)$, where $\sigma \in W(\mathfrak{g})$ is the longest word in the Weyl group of \mathfrak{g} . For $\mathfrak{g} = \mathfrak{sl}_2$, one has $\alpha^* = \alpha$.)

Use of the factorization rules depends on the one, two, and three point fusion rules, that is, the ranks of conformal blocks bundles for $n = 1, 2$, and 3. These are well-known for $n = 1$ and 2 for arbitrary \mathfrak{g} and for $n = 3$ when $\mathfrak{g} = \mathfrak{sl}_2$; see for instance [6, Lemma 4.2, Cor. 4.4]. We state them for \mathfrak{sl}_2 below.

Proposition 3.5 (Fusion rules for \mathfrak{sl}_2). Write $r_{\vec{\lambda}} = \text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\lambda}))$.

If $n = 1$:

$$r_{(\lambda)} = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 2$:

$$r_{(\lambda_1, \lambda_2)} = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 3$:

$$r_{(\lambda_1, \lambda_2, \lambda_3)} = \begin{cases} 1 & \text{if } \sum_{i=1}^3 \lambda_i \equiv 0 \pmod{2}, \sum_{i=1}^3 \lambda_i \leq 2\ell, \text{ and } \lambda_i \leq \frac{1}{2} \sum_{i=1}^3 \lambda_i. \\ 0 & \text{otherwise.} \end{cases}$$

We now present four small technical results that are used throughout the paper.

Lemma 3.6 (Odd Sum Rule). Let $\mathfrak{g} = \mathfrak{sl}_2$, let ℓ be an arbitrary level, and let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$. If $\sum_{i=1}^n \lambda_i$ is odd, then $r_{\vec{\lambda}} = 0$.

Proof. We argue by induction on n . The cases $n = 1, 2$, and 3 follow from the fusion rules for \mathfrak{sl}_2 (see Proposition 3.5 above). So suppose $n \geq 4$ and that the statement is true for $n - 1$. We can find two weights $a = \lambda_i$ and $b = \lambda_j$ in $\vec{\lambda}$ such that $a + b$ is even. Since the rank $r_{\vec{\lambda}}$ is symmetric with respect to permutations of the λ_i , we may assume without loss of generality that $\{i, j\} = \{1, 2\}$. Let $\vec{\mu} = (\lambda_3, \dots, \lambda_n)$ be complementary vector. Apply the factorization formula with the partition $\vec{\lambda} = (a, b) \cup \vec{\mu}$:

$$(3.7) \quad r_{\vec{\lambda}} = \sum_{\alpha=0}^{\ell} r_{(a,b,\alpha)} r_{\vec{\mu} \cup \alpha}.$$

Since $a + b$ is even, whenever α is odd, we have $r_{(a,b,\alpha)} = 0$ by the three point fusion rules. We also know that $\sum_{i=3}^n \lambda_i$ is odd, since $\sum_{i=1}^n \lambda_i$ is odd and $a + b$ is even. But then, by induction, $r_{\vec{\mu} \cup \alpha} = 0$ whenever α is even. Therefore all the summands in (3.7) are zero, and $r_{\vec{\lambda}} = \text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\lambda})) = 0$, as claimed. \square

Lemma 3.8 (Generalized Triangle Inequality). Let $\mathfrak{g} = \mathfrak{sl}_2$, let ℓ be an arbitrary level, and let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$. If there exists $i \in \{1, \dots, n\}$ such that $\lambda_i > \sum_{j \neq i} \lambda_j$, then $r_{\vec{\lambda}} = \text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\lambda})) = 0$.

Proof. This can be proved using factorization and induction, with $n = 3$ as the base case. \square

Lemma 3.9. Let $\vec{\mu} = (\mu_1, \mu_2, 1, 1)$ be a vector of weights for \mathfrak{sl}_2 satisfying $0 \leq \mu_i \leq \ell$ for $i = 1, 2$. Then

$$(3.10) \quad \text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu})) = \begin{cases} 2 & \text{if } \mu_1 = \mu_2 \text{ and } \mu_1 \notin \{0, \ell\} \\ 1 & \text{if } \mu_1 = \mu_2 \text{ and } \mu_1 \in \{0, \ell\} \\ 1 & \text{if } \mu_2 = \mu_1 \pm 2. \\ 0 & \text{otherwise} \end{cases}$$

Proof. First, we consider $\text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}))$. By factorization applied to the partition $\vec{\mu} = (\mu_1, \mu_2) \cup (1, 1)$ we obtain

$$(3.11) \quad r_{\vec{\mu}} = \sum_{\alpha=0}^{\ell} r_{(\mu_1, \mu_2, \alpha)} r_{(1, 1, \alpha)}.$$

By the two and three point fusion rules for \mathfrak{sl}_2 , we have:

$$\begin{aligned} r_{(1, 1, 0)} &= 1 \\ r_{(1, 1, 1)} &= 0 \\ r_{(1, 1, 2)} &= 1 \\ r_{(1, 1, \alpha)} &= 0, \text{ if } \alpha > 2. \end{aligned}$$

Thus only two summands on the right hand side of (3.11) are possibly nonzero. We have:

$$r_{\vec{\mu}} = r_{(\mu_1, \mu_2, 0)} r_{(1, 1, 0)} + r_{(\mu_1, \mu_2, 2)} r_{(1, 1, 2)} = r_{(\mu_1, \mu_2)} + r_{(\mu_1, \mu_2, 2)}.$$

By the two point fusion rules for \mathfrak{sl}_2 , $r_{(\mu_1, \mu_2)} = 0$ unless $\mu_1 = \mu_2$, and then this rank is 1. By the three point fusion rules, we have $r_{(\mu_1, \mu_2, 2)} = 0$ unless $\mu_2 \in \{\mu_1 - 2, \mu_1, \mu_1 + 2\}$. Also, $r_{(0, 0, 2)} = 0$, and $r_{(\ell, \ell, 2)} = \text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, (\ell, \ell, 2))) = 0$. The result follows. \square

Lemma 3.12. *Let $\vec{\mu} = (\mu_1, \mu_2, 1, 1)$ such that for $i \in \{1, 2\}$, $\mu_i \in \mathbb{Z}$ and $0 \leq \mu_i \leq \ell$. Consider the CB vector bundle $\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu})$ on $\overline{M}_{0,4} \cong \mathbb{P}^1$. Then*

$$(3.13) \quad \deg \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}) = \begin{cases} 0 & \text{if } \vec{\mu} \neq (\ell, \ell, 1, 1) \\ 1 & \text{if } \vec{\mu} = (\ell, \ell, 1, 1) \end{cases}.$$

Proof. We use [11, Cor.3.4, Formula (3.9)], with the notation $\text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\lambda})) = r_{\vec{\lambda}}$.

$$(3.14) \quad \begin{aligned} \deg \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}) &= \frac{1}{2(\ell + h^\vee)} \left\{ \left\{ r_{\vec{\mu}} \sum_{i=1}^4 c(\mu_i) \right\} - \right. \\ &\quad \left. \left\{ \sum_{\alpha=0}^{\ell} c(\alpha) \left\{ r_{(\mu_1, \mu_2, \alpha)} r_{(\mu_3, \mu_4, \alpha)} + r_{(\mu_1, \mu_3, \alpha)} r_{(\mu_2, \mu_4, \alpha)} + r_{(\mu_1, \mu_4, \alpha)} r_{(\mu_2, \mu_3, \alpha)} \right\} \right\} \right\}. \end{aligned}$$

Here $c(\alpha)$ denotes the Casimir scalar associated to α , and h^\vee denotes the dual Coxeter number of \mathfrak{sl}_2 . For \mathfrak{sl}_2 , these are given by $c(\alpha) = \alpha^2/2 + \alpha$, and $h^\vee = 2$.

We only want to compute $\deg \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu})$ in cases where $r_{\vec{\mu}} = \text{rank } \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}) > 0$. These cases are listed in the previous lemma. Let's check for instance that if $\mu_1 = \mu_2 = \mu, \mu \notin \{0, \ell\}$, then $\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}) = 0$:

Recall that the three point ranks for \mathfrak{sl}_2 are always 0 or 1. Note that $r_{(\mu_3, \mu_4, \alpha)} = 0$ unless $\alpha = 0, 2$. However, $c(0) = 0$, so the first term of the second line of Fakhruddin's formula contributes only when $\alpha = 2$. By the symmetry of the vector of weights μ , the second two terms of the second line are the same, and $r_{(\mu, 1, \alpha)} = 0$ unless $\alpha = \mu \pm 1$. Then we get

$$\begin{aligned} \deg \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}) &= \frac{1}{2(\ell + h^\vee)} \left\{ \left\{ r_{\vec{\mu}} \sum_{i=1}^4 c(\mu_i) - c(2) - 2c(\mu - 1) - 2c(\mu + 1) \right\} \right\} \\ &= \frac{1}{2(\ell + h^\vee)} \{ \{ 2(2c(\mu) + 2c(1)) - c(2) - 2c(\mu - 1) - 2c(\mu + 1) \} \} \\ &= 0. \end{aligned}$$

The other cases from Lemma 3.9 where $r_{\vec{\mu}} > 0$ can be checked similarly, yielding the result. \square

Remark 3.15. *Necessary and sufficient conditions for an \mathfrak{sl}_2 conformal blocks bundle to have $\text{rk}(\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\lambda})) \neq 0$ are given as follows.*

Lemma 3.16 (Swinarski, 2010). *Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $r_{\vec{\lambda}} \neq 0$ if and only if $\Lambda = \sum_{i=1}^n \lambda_i$ is even, and for any subset $I \subseteq \{1, \dots, n\}$ with $n - |I|$ odd, the inequality*

$$(3.17) \quad \Lambda - (n - |I| - 1)\ell \leq 2 \sum_{i \in I} \lambda_i$$

is satisfied.

The main achievement of this lemma is in finding the correct statement; with this in hand, the result may be proved in a straightforward way using induction on n and factorization. We omit the proof, as we do not use this result in the sequel.

4. INTERSECTING THE DIVISORS $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ WITH F-CURVES

In this section, in Theorem 4.2, we give a simple formula for the intersection of the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ with a basis of 1-cycles given by the first family of curves defined in Proposition 1.

Definition 4.1. Suppose that n is even and put

$$r_\ell(j, t) = \text{rank } \mathbb{V}(\mathfrak{sl}_2, \ell, \underbrace{(1, \dots, 1, t)}_{j \text{ times}}).$$

Theorem 4.2. $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2,i,1} = r_\ell(i, \ell) \cdot r_\ell(n-i-2, \ell).$

Proof. We write $P_\ell = \{0, 1, \dots, \ell\}$, and write $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ for elements of P_ℓ^4 . We use [11, Prop. 2.5] applied to $\mathbb{V}(\mathfrak{sl}_2, \ell, (1, \dots, 1))$ and the symmetric F-curve $F = F_{j_1, j_2, j_3, j_4}$, given by a partition $n = j_1 + j_2 + j_3 + j_4$.

Using the notation from Definition 4.1, this says:

$$(4.3) \quad \deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (1, \dots, 1))|_F) = \sum_{\vec{\mu} \in P_\ell^4} \deg \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}) \prod_{k=1}^4 r_\ell(j_k, \mu_k).$$

Recall that the two point fusion rules for \mathfrak{sl}_2 imply that $r_{(a,b)} = 0$ unless $a = b$, in which case $r_{(a,b)} = 1$. Since our F-curves have two 1's on the spine, the only nonzero summands in 4.3 occur when $\mu_3 = \mu_4 = 1$. By Lemma 3.12, we have $\deg \mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu}) = 0$ if $\vec{\mu} \neq (\ell, \ell, 1, 1)$, and 1 otherwise. The formula follows. \square

We present an example, which suggests several corollaries to Theorem 4.2.

Example 4.4. Consider the matrix of intersection numbers $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2,i,1}$ for $n = 16$, where in the table we put $\mathbb{D}_\ell^{\mathfrak{sl}_2}$ for $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$.

	$\mathbb{D}_1^{\mathfrak{sl}_2}$	$\mathbb{D}_2^{\mathfrak{sl}_2}$	$\mathbb{D}_3^{\mathfrak{sl}_2}$	$\mathbb{D}_4^{\mathfrak{sl}_2}$	$\mathbb{D}_5^{\mathfrak{sl}_2}$	$\mathbb{D}_6^{\mathfrak{sl}_2}$	$\mathbb{D}_7^{\mathfrak{sl}_2}$
$F_{1,1,1}$	1	0	0	0	0	0	0
$F_{1,1,2}$	0	32	0	0	0	0	0
$F_{1,1,3}$	1	0	55	0	0	0	0
$F_{1,1,4}$	0	32	0	40	0	0	0
$F_{1,1,5}$	1	0	63	0	19	0	0
$F_{1,1,6}$	0	32	0	52	0	6	0
$F_{1,1,7}$	1	0	64	0	25	0	1

Note that this matrix has full rank. This shows that the divisors are independent. Moreover, since in all of the columns there are curves that intersect the CB divisors in degree zero, this also shows that the divisors lie on the boundary of the nef cone.

We now derive six corollaries to Theorem 4.2. The first of these corollaries describes the pattern of zeroes observed in the matrix of the example above.

Corollary 4.5 (Vanishing intersecting numbers). (1) If $i < \ell$, then $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2,i,1} = 0$.

(2) If $i \not\equiv \ell \pmod{2}$, then $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2,i,1} = 0$.

Proof. For $i < \ell$, use the Generalized Triangle Inequality (Lemma 3.8), and for $i \geq \ell$, use the Odd Sum Rule (Lemma 3.6). \square

In the next four corollaries, we find formulas for intersection numbers of four of the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ (that is, formulas for the first two and last two columns of the matrix shown in the example above). First, we give a lemma computing certain ranks:

Lemma 4.6. (1) Suppose $\ell = 1$. Then for $k \in \mathbb{Z}, k \geq 0$ we have $r_1(2k+1, 1) = 1$.
 (2) Suppose $\ell = 2$. Then for $k \in \mathbb{Z}, k \geq 1$ we have $r_2(2k, 2) = 2^{k-1}$.
 (3) Suppose $\ell = 2$. Then for $k \in \mathbb{Z}, k \geq 0$ we have $r_2(2k+1, 1) = 2^k$.

Proof. We use induction on k and factorization.

For the first formula, by the two point fusion rules, $r_1(1, 1) = 1$. So suppose the formula is true up to $k-1$. Factorization and applying the Odd Sum Lemma yields

$$r_1(2k+1, 1) = r_1(2k, 0)r_1(2, 0) + r_1(2k, 1)r_1(2, 1) = r_1(2(k-1) + 1, 1) = 1.$$

For the second two formulas: we may check that $r_2(2, 2) = 1$ and $r_1(1, 1) = 1$. So suppose these two formulas work up to $k-1$. Factorization and applying the Odd Sum Lemma yields

$$r_2(2k, 2) = \sum_{\mu=0}^2 r_2(2k-1, \mu)r_{(1,2,\mu)} = r_2(2k-1, 1) = r_2(2(k-1) + 1, 1) = 2^{k-1}$$

and

$$r_2(2k+1, 1) = r_2(2(k-1) + 1, 1)r_{(1,1,0)} + r_2(2k, 2)r_{(1,1,2)} = 2^{k-1} + 2^{k-1} = 2^k.$$

□

Corollary 4.7.

$$(4.8) \quad \mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{a,b,c,d} = \begin{cases} 1 & abcd \text{ odd;} \\ 0 & abcd \text{ even.} \end{cases}$$

Proof. Suppose first that $abcd$ is even. Then at least one of the four integers, say a , is even. Then when we apply formula (4.3) to compute $\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{a,b,c,d}$, to get a nonzero summand, we must have $\mu_1 = 0$ to have $r_\ell(a, \mu_1) \neq 0$. (By the Odd Sum Lemma, we need μ_1 even, but $P_\ell = \{0, 1\}$ since $\ell = 1$.) Since $\mu_1 = 0$, by propagation, we know $\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu})$ is a pullback from $\overline{M}_{0,3} = pt$. Hence $\deg \mathbb{V}(\mathfrak{sl}_2, 1, \vec{\mu}) = 0$.

Now suppose that $abcd$ is odd. Then the only nonzero summand in formula (4.3) occurs when $\vec{\mu} = (1, 1, 1, 1)$. We can compute $\deg \mathbb{V}(\mathfrak{sl}_2, 1, (1, 1, 1, 1)) = 1$, and by Lemma 4.6 above, $r_1(a, 1)r_1(b, 1)r_1(c, 1)r_1(d, 1) = 1$. □

Corollary 4.9.

$$(4.10) \quad \mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{a,b,c,d} = \begin{cases} 0 & abcd \text{ odd;} \\ 2^{g-2} & abcd \text{ even.} \end{cases}$$

Proof. Suppose that $abcd$ is odd. If any μ_i is even, then $r_\ell(a, \mu_i) = 0$ by the Odd Sum Lemma. We have $P_\ell = \{0, 1, 2\}$ since $\ell = 2$, so we only possibly get a nonzero summand in formula (4.3) when $\vec{\mu} = (1, 1, 1, 1)$. We can compute $\deg \mathbb{V}(\mathfrak{sl}_2, 2, (1, 1, 1, 1)) = 0$, so in fact this summand is zero, too.

Suppose next that a, b are even while c, d are odd. To get a nonzero summand in formula (4.3) we must have μ_1 and μ_2 even and μ_3 and μ_4 odd. However, if μ_1 or μ_2 is zero, then by propagation, we know $\mathbb{V}(\mathfrak{sl}_2, \ell, \vec{\mu})$ is a pullback from $\overline{M}_{0,3} = pt$, and hence $\deg \mathbb{V}(\mathfrak{sl}_2, 1, \vec{\mu}) = 0$. Thus, we only get a nonzero summand in formula (4.3) when $\vec{\mu} = (2, 2, 1, 1)$. We compute $\deg \mathbb{V}(\mathfrak{sl}_2, 2, (2, 2, 1, 1)) = 1$, and use Lemma 4.6 to show that

$$r_2(a, 2)r_2(b, 2)r_2(c, 2)r_2(d, 2) = 2^{\frac{a}{2}-1}2^{\frac{b}{2}-1}2^{\frac{c-1}{2}}2^{\frac{d-1}{2}} = 2^{\frac{a+b+c+d}{2}-3} = 2^{g-2}.$$

Finally suppose that a, b, c, d are all even. As above, we may argue that we only get a nonzero summand in formula (4.3) when $\vec{\mu} = (2, 2, 2, 2)$, and then $\deg \mathbb{V}(\mathfrak{sl}_2, 2, (2, 2, 2, 2)) = 2$. We then use Lemma 4.6.ii to show that

$$2r_2(a, 2)r_2(b, 2)r_2(c, 2)r_2(d, 2) = 2 \cdot 2^{\frac{a}{2}-1}2^{\frac{b}{2}-1}2^{\frac{c}{2}-1}2^{\frac{d}{2}-1} = 2^{g-2}.$$

□

Lemma 4.11. (1) $r_\ell(k, k) = 1$, for all $1 \leq k \leq \ell$.

- (2) $r_\ell(k, k-2) = k-1$, for all $2 \leq k \leq \ell+1$.
 (3) $r_\ell(\ell, \ell+2) = \ell$.

Proof. For the first statement, we use induction on k with base case $k=1$. Indeed, $r_\ell(1,1)=1$ by the 2-point fusion rule. Assume $r_\ell(j,j)=1$ for $j < k$ and apply factorization with the partition $1^{k-1}|(1,k)$ to get:

$$r_\ell(k, k) = \sum_{0 \leq \mu \leq \ell} r_\ell(k-1, \mu) r_{(1,k,\mu)}.$$

By the three point fusion rules, we have $r_{(1,k,\mu)} = 0$ if $\mu < k-1$, or if $\mu > k+1$, or if $\mu = k$. We also have $r_{(1,k,\mu)} = 1$ if $\mu = k-1$ or $\mu = k+1$. However, if $\mu = k+1$, then $r_\ell(k-1, k+1) = 0$ by the Generalized Triangle Inequality (Lemma 3.8). So the only nonzero summand in $r_\ell(k, k)$ is $r_\ell(k-1, k-1)$, which is 1 by the induction hypothesis, and so we are done.

For the second statement, we use induction on k with base case $k=2$. When $k=2$, the statement is that $r_\ell(1,1,0)=1$, which is true by propagation and the two point fusion rules. So assume $r_\ell(j, j-2) = j-1$ for $2 \leq j \leq k-1$. Apply factorization to $1^{k-1} \cup (1, k-2)$.

$$r_\ell(k, k-2) = \sum_{0 \leq \mu \leq \ell} r_\ell(k-1, \mu) r_{(1,k-2,\mu)}.$$

As before, by the three point fusion rules, we have $r_{(1,k-2,\mu)} = 0$ if $\mu < k-3$, or if $\mu > k-1$, or if $\mu = k-2$. We also have $r_{(1,k-2,\mu)} = 1$ if $\mu = k-3$ or if $\mu = k-1$. Thus there are only two nonzero summands in $r_\ell(k, k-2)$:

$$r_\ell(k, k-2) = r_\ell(k-1, k-3) + r_\ell(k-1, k-1).$$

By the induction hypothesis, we have $r_\ell(k-1, k-3) = k-2$, and by the first statement of this lemma, we have $r_\ell(k-1, k-1) = 1$. Thus $r_\ell(k, k-2) = k-1$, as claimed.

For the third statement, we apply factorization using the partition $1^{\ell+1} \cup (1, \ell)$:

$$r_\ell(\ell+2, \ell) = \sum_{0 \leq \mu \leq \ell} r_\ell(\ell+1, \mu) r_{(1,\ell,\mu)}.$$

We can argue as we did above that there is only one nonzero summand, and it occurs for $\mu = \ell-1$. Thus $r_\ell(\ell+2, \ell) = r_\ell(\ell+1, \ell-1)$, and by the second statement, this is ℓ . \square

Corollary 4.12.

$$(4.13) \quad \mathbb{D}_{g-1, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2, i, 1, 1} = \begin{cases} 0 & i \neq g-1 \\ g-1 & i = g-1. \end{cases}$$

Proof. If $i \leq g-2$, then by Corollary 4.5, we have $\mathbb{D}_{g-1, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2, i, 1, 1} = 0$. And for $i = g$, by Theorem 4.2

$$\mathbb{D}_{g-1, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{g, g, 1, 1} = r_{g-1}(g, g-1) \cdot r_{g-1}(g, g-1).$$

As $2g-1$ is odd, by the Odd Sum Lemma 3.6, $r_{g-1}(g, g-1) = 0$. This leaves $i = g-1$. Again by Theorem 4.2

$$\mathbb{D}_{g-1, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{g+1, g-1, 1, 1} = r_{g-1}(g-1, g-1) \cdot r_{g-1}(g+1, g-1).$$

By the first statement of Lemma 4.11 we have $r_{g-1}(g-1, g-1) = 1$, and by the third statement we have $r_{g-1}(g+1, g-1) = g-1$. The result follows. \square

Corollary 4.14.

$$(4.15) \quad \mathbb{D}_{g, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2, i, 1, 1} = \begin{cases} 0 & i \leq g-1 \\ 1 & i = g. \end{cases}$$

Proof. If $i \leq g-1$, then by Corollary 4.5, we have $\mathbb{D}_{g,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2,i,1,1} = 0$. And for $i = g$, by Theorem 4.2

$$\mathbb{D}_{g,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{g,g,1,1} = r_g(g, g) \cdot r_g(g, g).$$

By the first statement of Lemma 4.11, we have $r_g(g, g) = 1$. \square

Corollary 4.16. $\{\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} : 1 \leq \ell \leq g\}$ is a basis for $\text{Pic}(\overline{M}_{0,2g+2}/S_{2g+2})$.

Proof. The matrix of intersection numbers between these divisors and the F-curves $F_{n-i-2,i,1,1}$ is lower triangular with nonzero entries on the diagonal, and so the divisors are linearly independent. To see this, note that by the Generalized Triangle Inequality, Lemma, one has $r_\ell(i, \ell) = 0$ if $i < \ell$. Thus, the entries above the diagonal are all zero. On the other hand, $r_\ell(\ell, \ell) = 1$. One can apply factorization to show that $r_\ell(n-i-2, \ell) \neq 0$ as well, by partitioning the weight vector $1^{n-i-2}\ell$ as $1^i\ell|1^{n-2i-2}$. As the rank of $\text{Pic}(\overline{M}_{0,n}/S_n)$ is g , the result follows. \square

5. CLASSES OF THE $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$

In this Section we prove Proposition 5.1, which gives a general formula for the classes of the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$. This expression depends on ranks of certain \mathfrak{sl}_2 CB-bundles. We give these ranks in Proposition 5.2, which is proved in Section 5.1. In Section 5.2 we give simplified expressions for the four elements of the family that generate extremal rays and we give simplified versions of the classes of some of the divisors that lie on higher dimensional extremal faces.

Recall that for $0 \leq t \leq \ell$

$$r_\ell(i, t) = \text{rank}(\mathbb{V}(\mathfrak{sl}_2, \ell, \underbrace{(1, \dots, 1)}_{i \text{ times}}, t)).$$

We will put $r_\ell(i, t) = 0$, for $t < 0$, and for $t > \ell$, and write

$$r_\ell(n) = \text{rank}(\mathbb{V}(\mathfrak{sl}_2, \ell, (1, \dots, 1))).$$

Proposition 5.1. $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} = \frac{1}{2(\ell+2)} \sum_{i=2}^{g+1} \left[\frac{i(n-i)}{n-1} \beta_1 - \beta_i \right] B_i$, where

$$\beta_1 = \frac{3}{2} r_\ell(n), \text{ and } \beta_i = \sum_{t=0}^{\ell} \left(\frac{t^2}{2} + t \right) r_\ell(i, t) r_\ell(n-i, t) \text{ for } 2 \leq i \leq g+1.$$

Proof. We obtain this formula from [11, Cor. 3.5], using that on \mathfrak{sl}_2 , one has $h^\vee = 2$ and $t^* = t$ and $c(t) = \frac{t^2}{2} + t$. \square

We give an explicit formula for the ranks $r_\ell(j, t)$ in the next proposition.

Proposition 5.2. Let $j \geq 0$, and let $0 \leq t \leq \ell$. Write $K := \lceil j/(2(\ell+2)) \rceil$. Then

(1) If j, t are both even, say $j = 2x$ and $t = 2y$, then

$$(5.3) \quad r_\ell(j, t) = \sum_{k=0}^K \left(b_k \binom{2x}{x-y-k(\ell+2)} - c_k \binom{2x}{x-(k+1)(\ell+2)+y+1} \right),$$

where $b_k = \frac{2y+2k(\ell+2)+1}{x+y+k(\ell+2)+1}$ and $c_k = \frac{(2k+2)(\ell+2)-2y-1}{x+(k+1)(\ell+2)-y}$; and

(2) If j, t are both odd, say $j = 2x+1$ and $t = 2y+1$, then

$$(5.4) \quad r_\ell(j, t) = \sum_{k=0}^K \left(b_k \binom{2x+1}{x-y-k(\ell+2)} - c_k \binom{2x+1}{x-(k+1)(\ell+2)+y+2} \right),$$

where $b_k = \frac{2y+2k(\ell+2)+2}{x+y+k(\ell+2)+2}$ and $c_k = \frac{2(k+1)(\ell+2)-2y-2}{x+(k+1)(\ell+2)-y}$.

(3) If j and t have opposite parity, then $r_\ell(j, t) = 0$.

Proposition 5.2 follows from Propositions 5.13 5.15. These are both proved in the following subsection.

Remark 5.5. Compare the Equations 5.3 and 5.4 above to the Verlinde formula (3.2). The formulas of Proposition 5.2 appear more complicated, but can be evaluated using only arithmetic operations (no trigonometric functions are required). Thus, it is clear from Proposition 5.2 that the ranks are rational numbers, which is not obvious in the Verlinde formula. In fact, with just a little more work, it is easy to argue that the numbers $r_\ell(j, t)$ in Proposition 5.2 are in fact integers.

5.1. Proofs of the rank formulas in Proposition 5.2. The proof of Proposition 5.2 involves three steps. First, in Proposition 5.6 we show the $r_\ell(j, t)$ are determined by a system of recurrences. Second, in Definition 5.12 and Proposition 5.13 we define a system of recurrences and solve it to get an array of numbers $r_\infty(j, t)$. Last, in Proposition 5.15 we explicitly relate the $\{r_\ell(j, t)\}$ and the $\{r_\infty(j, t)\}$.

Proposition 5.6. The ranks $r_\ell(j, t)$ are determined by the system of recurrences

$$(5.7) \quad r_\ell(j, t) = r_\ell(j-1, t-1) + r_\ell(j-1, t+1), \quad t = 1, \dots, \ell.$$

together with seeds

$$r_\ell(j, j) = 1, \text{ if } j \leq \ell, \text{ and } r_\ell(j, j) = 0, \text{ if } j > \ell.$$

Remark 5.8. We observe that (5.7) is somewhat reminiscent of the recurrence for Pascal's triangle.

Proof. Partition the weight vector $(1, \dots, 1, t) = 1^j t$ as $1^{j-1} \cup (1, t)$. If $j+t$ is odd, then by the Odd Sum Rule, Lemma 3.6, $r_\ell(j, t) = 0$. So assume $j+t$ is even. Then the factorization formula states

$$(5.9) \quad r_\ell(j, t) = \sum_{\mu=0}^{\ell} r_{(1^{j-1} \cup \mu)} r_{(1, t, \mu)}.$$

We can simplify this expression. Recall that by the \mathfrak{sl}_2 fusion rules (Prop. 3.5), $r_{(1, t, \mu)}$ is 0 if $\mu > t+1$ or if $\mu < t-1$. Thus the only possibly nonzero summands in (5.9) are when $\mu = t-1$, t , or $t+1$. But when $\mu = t$, by the Odd Sum Rule, Lemma 3.6, we have $r_{(1, t, t)} = 0$. Thus (5.9) simplifies to the following:

$$(5.10) \quad r_\ell(j, t) = r_\ell(j-1, t-1) + r_\ell(j-1, t+1) \quad t = 1, \dots, \ell-1;$$

$$(5.11) \quad r_\ell(j, \ell) = r_\ell(j-1, \ell-1).$$

Since $r_\ell(j-1, \ell+1) = 0$, we can unify the two lines above, yielding (5.7). □

Definition 5.12. Let $r_\infty(j, t)$ be the solutions of the system of recurrences

$$r_\infty(j, t) = r_\infty(j-1, t-1) + r_\infty(j-1, t+1)$$

with seeds

- (1) $r_\infty(j, t) = 0$ if $t > j$;
- (2) $r_\infty(j, -1) = 0$ for all j ; and
- (3) $r_\infty(j, j) = 1$ for all j .

Proposition 5.13. [Values of $r_\infty(j, t)$] Suppose $j \geq 0$ and $0 \leq t \leq j$.

- (1) If j and t are both even, say $j = 2x$ and $t = 2y$, then $r_\infty(j, t) = \frac{2y+1}{x+y+1} \binom{2x}{x-y}$.
- (2) If j and t are both odd, say $j = 2x+1$ and $t = 2y+1$, then $r_\infty(j, t) = \frac{2y+2}{x+y+2} \binom{2x+1}{x-y}$.
- (3) If $j \not\equiv t \pmod{2}$ then $r_\infty(j, t) = 0$.

Proof. Straightforward check. □

Remark 5.14. For $t = 1$, the formulas above give $\frac{1}{x+1}\binom{2x}{x}$. These are the well-known Catalan numbers.

Next we relate the numbers $\{r_\infty(j, t)\}$ and $\{r_\ell(j, t)\}$.

Proposition 5.15. Let $j \geq 0$, and let $0 \leq t \leq \ell$. Write $K := \lceil j/(2(\ell + 2)) \rceil$. Then

$$(5.16) \quad r_\ell(j, t) = \sum_{k=0}^K \left(r_\infty(j, t + 2(\ell + 2)k) - r_\infty(j, (2k + 2)(\ell + 2) - t - 2) \right).$$

In the formulas above, we use the convention that $\binom{n}{k} = 0$ if $k < 0$.

Corollary 5.17. Fix j, t . For ℓ sufficiently large with respect to j, t , we have $r_\ell(j, t) = r_\infty(j, t)$.

Proof. When ℓ is large, there are no nontrivial reflections in the algorithm presented below. \square

Proof of Proposition 5.15. Suppose that $j_0 \geq 0$, and $0 \leq t_0 \leq \ell$. To obtain $r_\ell(j_0, t_0)$ from the numbers $\{r_\infty(j, t)\}$:

- (1) Mark every $(\ell + 2^{\text{th}})$ column in the array of numbers $\{r_\infty(j, t)\}$. That is, starting with $k = 0$, while $(k + 1)(\ell + 2) - 1 \leq j$, mark the columns $t = (k + 1)(\ell + 2) - 1$.
- (2) Find the successive reflections of $r_\infty(j_0, t_0)$ across these marked columns.
- (3) $r_\ell(j_0, t_0)$ is the alternating sum of these reflections.

To prove this algorithm works, we use induction on j . For $j = 0$, we have $r_\infty(0, 0) = 1$ and $r_\infty(0, t) = 0$ for $t > 0$, so the algorithm above is correct. Suppose the algorithm works for all rows up to row $j - 1$. We check that it works for row j as well: run the algorithm on $r_\ell(j, t)$ to get $r_\ell(j, t)$ as an alternating sum of $r_\infty(j, t)$. Apply the recursion $r_\infty(j, t) = r_\infty(j - 1, t - 1) + r_\infty(j - 1, t + 1)$ to each term in the alternating sum. We can regroup the resulting terms and apply the induction hypothesis to write this expression as $r_\ell(j - 1, t - 1) + r_\ell(j - 1, t + 1)$. Thus the algorithm produces numbers that satisfy the correct recurrences.

The algorithm doesn't apply to the zero seed values. (That is, we need to define these values separately, so we may set them to zero as desired.) We only have to check the following seeds: for $j \leq \ell$, there are no nontrivial reflections, and we have $r_\ell(j, j) = r_\infty(j, j) = 1$. \square

Proposition 5.2 now follows from Propositions 5.13 and 5.15.

Example 5.18. By using propagation, factorization, and recursion, we compute $r_3(15, 3) = 377$. We compute $r_3(15, 3)$ using the proposition. A portion of the matrix $r_\infty(i, j)$ is shown in the table below, and we see that $r_\infty(15, 3) - r_\infty(15, 5) + r_\infty(15, 13) - r_\infty(15, 15) = 2002 - 1638 + 14 - 1 = 377$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1	0	1														
2	1	0	1													
3	0	2	0	1												
4	2	0	3	0	1											
5	0	5	0	4	0	1										
6	5	0	9	0	5	0	1									
7	0	14	0	14	0	6	0	1								
8	14	0	28	0	20	0	7	0	1							
9	0	42	0	48	0	27	0	8	0	1						
10	42	0	90	0	75	0	35	0	9	0	1					
11	0	132	0	165	0	110	0	44	0	10	0	1				
12	132	0	297	0	275	0	154	0	54	0	11	0	1			
13	0	429	0	572	0	429	0	208	0	65	0	12	0	1		
14	429	0	1001	0	1001	0	637	0	273	0	77	0	13	0	1	
15	0	1430	0	2002	0	1638	0	910	0	350	0	90	0	14	0	1

5.2. Simplified versions of the four extremal rays and four other extremal divisors. In Section 6.1 we show that for $\ell \in \{1, 2, g-1, g\}$ the divisors $\mathbb{D}_{\ell, (1, \dots, 1)}^{\mathfrak{sl}_2}$ generate extremal rays of the symmetric nef cone. We give expressions for their divisor classes now.

Theorem 5.19.

$$\begin{aligned} \mathbb{D}_{1, (1, \dots, 1)}^{\mathfrak{sl}_2} &= \sum_{2 \leq k \leq g+1, k \text{ even}} \frac{k(n-k)}{4(n-1)} B_k + \sum_{2 \leq k \leq g+1, k \text{ odd}} \frac{(k-1)(n-k-1)}{4(n-1)} B_k; \\ \mathbb{D}_{2, (1, \dots, 1)}^{\mathfrak{sl}_2} &= 3 \cdot 2^{g-1} \left(\sum_{2 \leq k \leq g+1, k \text{ even}} \left(\frac{k(n-k)}{8(n-1)} - \frac{1}{6} \right) B_k + \sum_{2 \leq k \leq g+1, k \text{ odd}} \frac{(k-1)(n-k-1)}{8(n-1)} B_k \right); \\ \mathbb{D}_{g-1, (1, \dots, 1)}^{\mathfrak{sl}_2} &= (g-1) \left(\sum_{k=2}^g \frac{(k-1)k}{(n-1)} B_k + \left(\frac{g^2 - g - 1}{(n-1)} \right) B_{g+1} \right); \\ \mathbb{D}_{g, (1, \dots, 1)}^{\mathfrak{sl}_2} &= 2 \sum_{k=2}^{g+1} \frac{(k-1)k}{(n-1)} B_k. \end{aligned}$$

Proof. We intersect the expressions above with the basis of 1-cycles $\{F_{1,1,i} : 1 \leq i \leq g\}$ and check that we get the formulas given in Corollaries 4.7, 4.9, 4.12, and 4.14.

Note the simple but useful formula which follows from [21]:

$$\sum_{k=2}^{n/2} c_k B_k \cdot F_{1,1,i} = -c_{i+2} - c_i + c_2 + 2c_{i+1}.$$

We consider $\mathbb{D}_{1, (1, \dots, 1)}^{\mathfrak{sl}_2}$ first. Multiplying by the constant $4(n-1)$ one has for i even, that

$$4(n-1) \mathbb{D}_{1, \{1, \dots, \}}^{\mathfrak{sl}_2} \cdot F_{1,1,i} = -(i+2)(n-(i+2)) - i(n-i) + 2(n-2) + 2i(n-i-2) = 0,$$

and for i odd,

$$4(n-1) \mathbb{D}_{1, \{1, \dots, \}}^{\mathfrak{sl}_2} \cdot F_{1,1,i} = -(i+1)(n-i-3) - (i-1)(n-i-1) + 2(n-2) + 2(i+1)(n-(i+1)) = 4(n-1).$$

In other words,

$$(5.20) \quad \mathbb{D}_{1, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2, i, 1, 1} = \begin{cases} 1 & i \text{ odd}; \\ 0 & i \text{ even}, \end{cases}$$

Therefore, by Corollary 4.16 and Lemma 4.7, we have the correct expression for $\mathbb{D}_{1, (1, \dots, 1)}^{\mathfrak{sl}_2}$.

Next, we consider $\mathbb{D}_{2, (1, \dots, 1)}^{\mathfrak{sl}_2}$. One checks easily that for i even,

$$\frac{1}{3} \cdot \frac{1}{2^{g-1}} \mathbb{D}_{2, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{1,1,i} = \frac{1}{6} \implies \mathbb{D}_{2, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{1,1,i} = 2^{g-2}.$$

It is also straightforward to check that for i odd, $\mathbb{D}_{2, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{1,1,i} = 0$. In other words,

$$(5.21) \quad \mathbb{D}_{2, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{n-i-2, i, 1, 1} = \begin{cases} 0 & i \text{ odd}; \\ 2^{g-2} & i \text{ even}, \end{cases}$$

Therefore, by Corollary 4.16 and Lemma 4.9, we have the correct expression for $\mathbb{D}_{2, (1, \dots, 1)}^{\mathfrak{sl}_2}$.

Next, we consider $\mathbb{D}_{g-1, (1, \dots, 1)}^{\mathfrak{sl}_2}$. For $i \leq g-2$,

$$\frac{(n-1)}{(g-1)} \mathbb{D}_{g-1, (1, \dots, 1)}^{\mathfrak{sl}_2} \cdot F_{1,1,i} = -(i+1)(i+2) - (i-1)i + 2 + 2i(i+1) = 0.$$

For $i = g$,

$$(5.22) \quad \frac{(n-1)}{(g-1)} \mathbb{D}_{g-1, (1, \dots, 1)}^{\text{sl}_2} \cdot F_{1,1,g,g} = -2c_g + c_2 + 2c_{g+1} \\ = -2g(g-1) + 2 + 2 \frac{(g^3 - 2g^2 + 1)}{(g-1)} = -2g(g-1) + g(2g-2) = 0.$$

Whereas, if $i = g-1$, then $i+1 = g$ and $i+2 = g+1$, and so

$$\frac{(n-1)}{(g-1)} \mathbb{D}_{g-1, (1, \dots, 1)}^{\text{sl}_2} \cdot F_{1,1,g-1} = -\frac{(g^3 - 2g^2 + 1)}{(g-1)} - (g-2)(g-1) + 2 + 2g(g-1) = 2g+1 \\ \Rightarrow \mathbb{D}_{g-1, (1, \dots, 1)}^{\text{sl}_2} \cdot F_{1,1,g-1} = g-1$$

Therefore, by Corollary 4.16 and Lemma 4.12, we have the correct expression for $\mathbb{D}_{g-1, (1, \dots, 1)}^{\text{sl}_2}$.

Finally, we consider $\mathbb{D}_{g, (1, \dots, 1)}^{\text{sl}_2}$. For $i \leq g-1$,

$$2(n-1) \mathbb{D}_{g, (1, \dots, 1)}^{\text{sl}_2} \cdot F_{1,1,i} = -(i+1)(i+2) - (i-1)i + 2 + 2i(i+1) = 0,$$

and for $i = g$,

$$2(n-1) \mathbb{D}_{g, (1, \dots, 1)}^{\text{sl}_2} \cdot F_{1,1,g} = -2g(g-1) + 2 + 2g(g+1) = 2(n-1).$$

Therefore, by Corollary 4.16 and Lemma 4.14, we have the correct expression for $\mathbb{D}_{g, (1, \dots, 1)}^{\text{sl}_2}$. \square

We have also found simple formulas for the classes of some of the other divisors by ad hoc methods. We list a few such formulas here.

Example 5.23. $\mathbb{D}_{3, (1, \dots, 1)}^{\text{sl}_2}$. Let F_n be the Fibonacci numbers given by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Then $\mathbb{D}_{3, (1, \dots, 1)}^{\text{sl}_2} = \sum b_k B_k$, where if $k = 2j+1$ is odd, then

$$b_k = \frac{1}{10} \left(\frac{3}{2} \frac{k(n-k)}{n-1} F_{2g+1} - \frac{3}{2} F_{2j+1} F_{2(g-j)+1} - \frac{15}{2} F_{2j} F_{2(g-j)} \right).$$

and if $k = 2j$ is even, then

$$b_k = \frac{1}{10} \left(\frac{6j(g-j+1)}{2g+1} F_{2g+1} - 4F_{2j} F_{2(g-j+1)} \right).$$

Example 5.24. $\mathbb{D}_{4, (1, \dots, 1)}^{\text{sl}_2} = \sum b_k B_k$, where if k is odd, then

$$b_k = \frac{1}{16} \left(\frac{k(n-k)}{n-1} (3^{\frac{n-2}{2}} + 1) - \frac{1}{2} (3^{\frac{k-1}{2}} + 1) (3^{\frac{n-k-1}{2}} + 1) - \frac{5}{2} (3^{\frac{k-1}{2}} - 1) (3^{\frac{n-k-1}{2}} - 1) \right)$$

and if k is even, then

$$b_k = \frac{1}{12} \left(\frac{k(n-k)}{n-1} \frac{3}{4} (3^{\frac{n-2}{2}} + 1) - 4 \cdot 3^{\frac{n-4}{2}} - 3(3^{\frac{k-2}{2}} - 1) (3^{\frac{n-k-2}{2}} - 1) \right)$$

Example 5.25. $\mathbb{D}_{g-2, (1, \dots, 1)}^{\text{sl}_2} = \sum b_k B_k$, where

$$b_k = \begin{cases} \frac{4(n-7)(n-2)(k-1)k}{16(n-1)} & \text{if } k \leq g-1 \\ \frac{n^4 - 17n^3 + 90n^2 - 152n + 96}{16(n-1)} & \text{if } k = g \\ \frac{n^4 - 15n^3 + 60n^2 - 20n - 32}{16(n-1)} & \text{if } k = g+1. \end{cases}$$

6. POSITION OF THE DIVISORS $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ IN THE NEF CONE

The divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ are semiample, and so (suitably large multiples of) these divisors define morphisms on $\overline{M}_{0,n}$. Any S_n -invariant divisor lies on the interior of the cone of effective divisors ([21] and [15]), and since the $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ are symmetric, the morphisms they define are birational. As the divisors are nef, we can tell more about the morphisms they define by finding their location in the nef cone. Base point free divisors in the interior of the nef cone give embeddings, and those on the boundary define contractions.

6.1. The $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ are extremal. In this section we show that the morphisms given by the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ on $\overline{M}_{0,n}/S_n$ are in fact birational contractions by proving the following.

Theorem 6.1. *The divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$, for $1 \leq \ell \leq g$ reside on the boundary of $\text{Nef}(\overline{M}_{0,n}/S_n)$, and for $\ell \in \{1, 2, g-1, g\}$ the divisors span extremal rays of the cone. In particular, the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$, for $1 \leq \ell \leq g$ define birational contractions on $\overline{M}_{0,n}/S_n$.*

In Section 5 we gave the classes of the divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$. As an application, we identify the corresponding morphisms, which we discuss in detail in Section 7.

Proof. We write $n = 2(g+1)$ and recall that the space $\text{Pic}(\overline{M}_{0,n}/S_n) \otimes \mathbb{R}$ is g -dimensional. Therefore, a divisor lies on a face of dimension k if it intersects $g-k$ basis elements in degree zero. Hence the divisor $\mathbb{D}_{\ell,\vec{\lambda}}^{\mathfrak{sl}_2}$ lies on a face of codimension at least ℓ by Corollary 4.5 using the basis $\{F_{1,1,i} : 1 \leq i \leq g\}$, given by Corollary 4.16. In particular by these Corollaries, $\mathbb{D}_{g-1,\vec{\lambda}}^{\mathfrak{sl}_2}$ and $\mathbb{D}_{g,\vec{\lambda}}^{\mathfrak{sl}_2}$ generate extremal rays of the cone. To show that $\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2}$ and $\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2}$ also generate extremal rays, we will use the two other sets of independent curves \mathcal{C}_2 and \mathcal{C}_3 from Theorem 2.1.

First, we note that by [11, Prop. 5.2], which we can apply since $\ell = 1$, one has $\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{a,b,c,d} = 0$ if $abcd \equiv 0 \pmod{2}$. In particular, $\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{2,2,i,n-4-i} = 0$ for all $1 \leq i \leq g-1$. Since these are the curves from \mathcal{C}_2 , and since $\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2}$ is nontrivial, we have that it spans an extremal ray of the symmetric nef cone.

Next, we show that $\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{a,b,c,d} = 0$ if $abcd \equiv 1 \pmod{2}$. This implies a, b, c, d are all odd. This is proved using [11, Prop. 2.5],

$$\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2} \cdot F_{a,b,c,d} = \sum_{\substack{\vec{\mu} = \{\mu_1, \dots, \mu_4\} \\ 0 \leq \mu_i \leq 2}} \deg(\mathbb{V}(\mathfrak{sl}_2, 2, \vec{\mu})) r_2(a, \mu_1) r_2(b, \mu_2) r_2(c, \mu_3) r_2(d, \mu_4).$$

Since the level is 2, each μ_i can only be 0, 1, or 2. If μ_i is even for any i , then the corresponding rank is 0, e.g. if μ_1 is even then $r_{a\mu_1} = 0$. It remains only to consider the case where $\mu_i = 1$ for $i = 1, \dots, 4$. But an explicit calculation using 3.14 shows that $\deg(\mathbb{V}(\mathfrak{sl}_2, 2, (1, 1, 1, 1))) = 0$. Thus, there are no nonzero contributions to Fakhruddin's formula 4.3. We show in Proposition 3 that for $g = 2k$ or $g = 2k-1$, the set $\mathcal{C}_3 = \{F_{3,3,2i+1} : 0 \leq i \leq k-2\} \cup \{F_{1,1,2i+1} : 0 \leq i \leq k-1\}$ consists of independent curves. Therefore $\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2}$ is extremal in the symmetric nef cone. \square

6.2. The $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ are not all log canonical. The Ray Theorem of Keel and McKernan [21, Thm. 1.2] and its extension by Farkas and Gibney [12, Thm. 4] gives a tool for detecting nef divisors in what we call the log canonical part of the cone.

Definition 6.2. We say a divisor D on $\overline{M}_{0,n}$ is log canonical if D may be expressed as an effective combination

$$D = c(K_{\overline{M}_{0,n}} + \sum_{I \subset \{1, \dots, n\}} c_I \delta_I),$$

where c and the c_I are any nonnegative rational numbers such that $0 \leq c_I \leq 1$ for all I .

If a divisor D is log canonical, and D intersects every F-curve nonnegatively, then D is nef by the Ray Theorem. Moreover, D is semiample by [5].

For symmetric divisors, we can say even more. Suppose D is S_n -equivariant. Then if D is also log canonical, it is actually possible to write D in the form

$$D = c(K_{\overline{M}_{0,n}} + \sum_{i=2}^{g+1} b_i B_i),$$

where $c > 0$ and $0 \leq b_i \leq 1$ for all i . This can be easily proved by averaging the log canonical expression for D over S_n . We call a divisor of the form $c(K_{\overline{M}_{0,n}} + \sum_{i=2}^{g+1} b_i B_i)$ with $0 \leq b_i \leq 1$ *symmetrically log canonical*.

Some of the divisors $\mathbb{D}_{\ell, (1, \dots, 1)}^{sl_2}$ are of this form, but others are not. We interpret the failure of these divisors to be symmetrically log canonical to mean that they are outside the part of $\text{Nef}(\overline{M}_{0,n})$ that can be understood combinatorially. This motivated us to study the geometry behind them in more detail.

Proposition 6.3. (1) $\mathbb{D}_{1, (1, \dots, 1)}^{sl_2}$ is symmetrically log canonical for $n = 6, 8, 10$, but not for $n \geq 12$.

(2) $\mathbb{D}_{2, (1, \dots, 1)}^{sl_2}$ is symmetrically log canonical for all $n \geq 6$.

(3) $\mathbb{D}_{g-1, (1, \dots, 1)}^{sl_2}$ is symmetrically log canonical for $n = 10, 12, 14$, but not for $n \geq 16$.

(4) $\mathbb{D}_{g, (1, \dots, 1)}^{sl_2}$ is symmetrically log canonical for $n = 8, 10, 12$, but not for $n \geq 14$.

Proof. The claims regarding the divisors $\mathbb{D}_{\ell, (1, \dots, 1)}^{sl_2}$ for small values of n may be verified by direct calculations.

The following identity shows that $\mathbb{D}_{2, (1, \dots, 1)}^{sl_2}$ is symmetrically log canonical for all n :

$$\frac{8}{3 \cdot 2^{g-1}} \mathbb{D}_{2, (1, \dots, 1)}^{sl_2} = K_{\overline{M}_{0,n}} + \frac{2}{3} \sum_{i \text{ even}} B_i + \sum_{i \text{ odd}} B_i.$$

It is easy to show that for $n \geq 12$, g odd, there is no triple (c, b_2, b_g) making $\mathbb{D}_{1, (1, \dots, 1)}^{sl_2}$ symmetrically log canonical, and $n \geq 12$, g even, there is no triple (c, b_2, b_{g+1}) making $\mathbb{D}_{1, (1, \dots, 1)}^{sl_2}$ symmetrically log canonical. Similarly, it is easy to show that for $n \geq 16$, there is no triple (c, b_3, b_g) making $\mathbb{D}_{g-1, (1, \dots, 1)}^{sl_2}$ symmetrically log canonical, and for $n \geq 14$, there is no triple (c, b_2, b_{g+1}) making $\mathbb{D}_{g, (1, \dots, 1)}^{sl_2}$ symmetrically log canonical. \square

7. MORPHISMS DEFINED BY THE DIVISORS $\mathbb{D}_{\ell, (1, \dots, 1)}^{sl_2}$

In this section we consider the morphisms defined by the extremal divisors and by divisors that lie on some faces spanned by them.

7.1. Levels 1 and 2 and extended Torelli maps. Let $h : \overline{M}_{0, 2(g+1)} \rightarrow \overline{M}_g$ be the morphism defined by taking $(C, \vec{p}) \in \overline{M}_{0, 2(g+1)}$ to the stable curve of genus g obtained by taking a double cover of C branched at the set marked points $\vec{p} = \{p_1, \dots, p_n\}$. In this section we will show that the divisor $\mathbb{D}_{1, (1, \dots, 1)}^{sl_2}$ defines a morphism that factors through h .

The following formula for the pullback of a divisor on \overline{M}_g along the map h will be useful:

Lemma 7.1. *Let $h : \overline{M}_{0,2(g+1)} \rightarrow \overline{H}_g \subset \overline{M}_g$ be the isomorphism onto the hyperelliptic locus in \overline{M}_g , and let $D = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i \delta_i$ be a divisor on \overline{M}_g . Then*

$$h^*(D) = \sum_{\substack{2 \leq k \leq \lfloor \frac{g}{2} \rfloor \\ k \text{ even}}} \left(\frac{ak(n-k)}{8(n-1)} - 2b_0 \right) B_k + \sum_{\substack{2 \leq k \leq \lfloor \frac{g}{2} \rfloor \\ k \text{ odd}}} \left(\frac{a(k-1)(n-k-1)}{8(n-1)} - \frac{b_i}{2} \right) B_k.$$

Proof. This follows from [8, p. 468-470 and Prop. 4.7]. \square

The classical Torelli map $M_g \xrightarrow{t} A_g$, which takes a smooth curve X of genus g to its Jacobian, extends to a regular map

$$t^{\text{Sat}} : \overline{M}_g \rightarrow \overline{A}_g^{\text{Sat}},$$

where $\overline{A}_g^{\text{Sat}}$ is the Satake compactification of the moduli space A_g . This morphism t^{Sat} is given by the divisor λ . In other words, $\lambda = (t^{\text{Sat}})^*(\Theta)$, where Θ is the theta divisor on $\overline{A}_g^{\text{Sat}}$ [2].

Theorem 7.2. *The divisor $\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2}$ defines the composition*

$$\overline{M}_{0,2g+2}/S_{2g+2} \xrightarrow{h} \overline{M}_g \xrightarrow{\tau^{\text{Sat}}} \overline{A}_g^{\text{Sat}}.$$

Proof. To prove this, we use Lemma 7.1 and Theorem 5.19 to show

$$\mathbb{D}_{1,(1,\dots,1)}^{\mathfrak{sl}_2} = 2h^*(\lambda).$$

Because λ is the semiample divisor that defines the morphism from \overline{M}_g to $\overline{A}_g^{\text{Sat}}$, the result follows. \square

It is natural to wonder whether any of the other divisors $\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2}$ define morphisms that factor through h . We present some evidence which suggests this may be true for $\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2}$.

Theorem 7.3. *Write*

$$h : \overline{M}_{0,2g+2}/S_{2g+2} \rightarrow \overline{H}_g \hookrightarrow \overline{M}_g.$$

Then

$$\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2} = \frac{1}{2}h^*(12\lambda - \delta_0).$$

Proof. Use Lemma 7.1 and Theorem 5.19. \square

Since $\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2}$ is a conformal blocks divisor (or, alternatively, because it is symmetrically log canonical), we know that $h^*(12\lambda - \delta_0)$ is semiample on $\overline{M}_{0,n}$. However, we do not know whether $12\lambda - \delta_0$ is semiample on \overline{M}_g in general:

Question 7.4. *Is $12\lambda - \delta_0$ semiample on \overline{M}_g for all $g \geq 4$?*

It is known that $12\lambda - \delta_0$ is nef [10, Prop. 3.3]. Also, Rulla shows that $12\lambda - \delta_0$ is base-point free for $g = 3$ [26, Prop. 2.3.6]. If $12\lambda - \delta_0$ is base-point free for all g , then Theorem 7.3 shows that the morphism given by $\mathbb{D}_{2,(1,\dots,1)}^{\mathfrak{sl}_2}$ factors through the hyperelliptic locus, as in the case for $g = 3$.

Let X denote the image of the linear system $|12\lambda - \delta_0|$. Even for $g = 3$, where we know this divisor defines a morphism, we don't know a modular interpretation or classical description of X . As one can see in [26, Fig. 2.8], X corresponds to a wall in the effective cone of \overline{M}_3 that lies between the full dimensional chambers that correspond to $\overline{A}_3^{\text{Vor}(2)}$ and $\overline{M}_3^{\text{ps}}$. It seems a reasonable guess that there might be morphisms from $\overline{A}_g^{\text{Vor}(2)}$ and $\overline{M}_g^{\text{ps}}$ to X that are small modifications. Here $\overline{M}_g^{\text{ps}}$ stands for the moduli space of pseudostable curves (cf. [27], [20], [17]).

7.2. Levels g and $g - 1$ and points on a line. In [29], Matthew Simpson identified certain log canonical models of $\overline{M}_{0,n}$ as Hassett weighted spaces $\overline{M}_{0,A}$ ([18]). This result has been extended and reproved several times ([1], [13], [22]). We use the notation of Kiem and Moon [22].

Let $(\mathbb{P}^1)^n // \mathrm{SL}(2)$ denote the GIT quotient with the symmetric linearization.

Theorem 7.5. *The extremal divisor $\mathbb{D}_{g,(1,\dots,1)}^{\mathrm{sl}_2}$ defines a sequence of contractions through Hassett's moduli spaces $\overline{M}_{0,n \cdot \epsilon}$ of weighted pointed stable curves with symmetric weights $n \cdot \epsilon = \{\epsilon, \dots, \epsilon\}$. Namely, for $n = 2g + 2$*

$$\overline{M}_{0,n} = \overline{M}_{0,n \cdot \epsilon_{g-1}} \rightarrow \overline{M}_{0,n \cdot \epsilon_{g-2}} \rightarrow \cdots \rightarrow \overline{M}_{0,n \cdot \epsilon_1} \rightarrow (\mathbb{P}^1)^n // \mathrm{SL}(2),$$

where $\frac{1}{g+2-k} < \epsilon_k < \frac{1}{g+1-k}$.

Proof. Compare the intersection numbers of $\mathbb{D}_{g,(1,\dots,1)}^{\mathrm{sl}_2}$ given in Cor. 4.14 with those for the pullback of the distinguished ample line bundle on the GIT quotient given in [1]. \square

We have $A = n \cdot \epsilon_1 = (\frac{2}{n-2}, \dots, \frac{2}{n-2})$. Kiem and Moon identify $\overline{M}_{0,n \cdot \epsilon_1}$ as Kirwan's partial desingularization of $(\mathbb{P}^1)^n // \mathrm{SL}(2)$. They give a description of $\mathrm{Pic}(\overline{M}_{0,n \cdot \epsilon_k})$ in [22, Thm 6.1.2], from which it follows that $\dim \mathrm{NS}(\overline{M}_{0,n \cdot \epsilon_1}/S_n) = 2$. Alternatively, one can easily check the following claim directly:

Claim 7.6. *Let $\rho_{n \cdot \epsilon_1} : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n \cdot \epsilon_1}$ be the birational contraction defined by Hassett.*

- (1) $\rho_{n \cdot \epsilon_1}$ contracts Δ_I if $|I| \neq 2, \frac{n}{2}$.
- (2) $\rho_{n \cdot \epsilon_1}$ contracts $F_{k,1,1}$ if $k \leq g - 2$.

By Theorem 7.5 above, we know that the morphism associated to $\mathbb{D}_{g,(1,\dots,1)}^{\mathrm{sl}_2}$ factors through $\overline{M}_{0,n \cdot \epsilon_1}$, and by [11, Prop. 4.7], we know that the morphism associated to $\mathbb{D}_{g-1,(1,\dots,1)}^{\mathrm{sl}_2}$ also factors through $\overline{M}_{0,n \cdot \epsilon_1}$. We combine this with the intersection numbers given in Corollaries 4.12 and 4.14 to obtain the following result:

Theorem 7.7. *Let Y_{g-1} denote the image of the morphism associated to the divisor $\mathbb{D}_{g-1,(1,\dots,1)}^{\mathrm{sl}_2}$. Then Y_{g-1} is the (generalized) flip of $(\mathbb{P}^1)^n // \mathrm{SL}(2)$ which contracts the curve $F_{g,1,1}$ in $\overline{M}_{0,n \cdot \epsilon_1}$:*

$$\begin{array}{ccc} & \overline{M}_{0,n} & \\ & \downarrow \rho_{n \cdot \epsilon_1} & \\ & \overline{M}_{0,n \cdot \epsilon_1} & \\ \swarrow \text{contract } F_{g-1,1,1} & & \searrow \text{contract } F_{g,1,1} \\ (\mathbb{P}^1)^n // \mathrm{SL}(2) & \text{-----} & Y_{g-1} \end{array}$$

One rich source of flips of GIT quotients is variation of GIT ([9], [28]). However, comparing the intersection numbers given in Corollary 4.12 with those given in [1] suggests that the flip described above does not arise by varying the linearization on $(\mathbb{P}^1)^n // \mathrm{SL}(2)$.

7.3. Finding nef divisors on $\overline{M}_{2(g+1)}$ using the flag morphism. Given any point $(E; p) \in \overline{M}_{1,1}$, let $f : \overline{M}_{0,n} \rightarrow \overline{M}_n$ be the morphism given taking $(C; p_1, \dots, p_n) \in \overline{M}_{0,n}$ to the curve of genus n obtained by attaching n copies of E to C by identifying p and p_i . We call this the *flag morphism*. The results of [16] establish a close connection between the properties of a divisor D on \overline{M}_n and its pullback along this morphism f^*D . For instance, every nef divisor on $\overline{M}_{0,n}/S_n$ is the pullback of a nef divisor on \overline{M}_n along this map ([16, Thm. 0.7]); thus, it is natural to ask which nef divisors on \overline{M}_n pull back to the divisors $\{\mathbb{D}_{\ell,(1,\dots,1)}^{\mathrm{sl}_2} : \ell \in \{1, 2, g-1, g\}\}$. But we can

obtain slightly stronger results using [16, Thm. 0.3], which says that a divisor D on \overline{M}_n which intersects all the F-curves (defined below) nonnegatively is nef if and only if the divisor f^*D is nef on $\overline{M}_{0,n}$. Thus, we can produce nef divisors on \overline{M}_n by finding F-divisors on \overline{M}_n which pull back to $\{\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} : \ell \in \{1, 2, g-1, g\}\}$. We carry this out in Proposition 7.10.

Throughout this paper we have been using F-curves on $\overline{M}_{0,n}$ (see Section 2.2 and the references there for a discussion). One can also define F-curves on $\overline{M}_{g,n}$. In [16] Theorem 2.2 and Figure 2.3, five types of F-curves on $\overline{M}_{g,n}$ are defined and pictured. We will refer to these as F-curves of types (1) – (5) in the sequel. For the reader's convenience, we state a combinatorial definition of an F-divisor D on $\overline{M}_{2(g+1)}$ from [16] which we rewrite slightly to fit our situation exactly.

Theorem 7.8 ([16, Thm. 2.1]). *Let $n = 2(g+1)$ and consider the divisor $D = a\lambda - \sum_{i=0}^{g+1} b_i \delta_i$ on $\overline{M}_{2(g+1)}$. Then D is an F-divisor if and only if it satisfies the following inequalities:*

- (1) $a - 12b_0 + b_1 \geq 0$;
- (2) $b_i \geq 0$;
- (3) $2b_0 - b_i \geq 0$;
- (4) $b_i + b_j \geq b_{i+j}$, for all $i, j \neq 0$;
- (5) $b_i + b_j + b_k + b_\ell \geq b_{i+j} + b_{i+k} + b_{i+\ell}$, for all $i, j, k, \ell \neq 0$, such that $i + j + k + \ell = 2(g+1)$.

Each of the inequalities (1) – (5) of Theorem 7.8 is satisfied by a divisor D as long as D nonnegatively intersects the corresponding F-curves of types (1) – (5).

In [16, Thm. 4.7] a nef divisor which we'll denote \mathcal{D} is defined with the property that \mathcal{D} strictly positively intersects the F-curve of types (1) – (4) while it intersects the F-curve of types (5) in degree zero. In particular, it is shown that $f^*\mathcal{D}$ is trivial. We will use this divisor \mathcal{D} in Proposition 7.10, and so for the reader's convenience, we recall its definition.

Definition 7.9. *On $\overline{M}_{2(g+1)}$ we consider the divisor*

$$\mathcal{D} = \alpha\lambda - \beta\delta_0 - \sum_{i=1}^{g+1} i(2(g+1) - i)\delta_i.$$

Theorem 1 ([16, Thm. 4.7]). *Let \mathcal{D} be the divisor from Definition 7.9. For any choice of α and β such that $\alpha > 12\beta - (2g+1)$, and $2\beta > (g+1)^2$:*

- (1) \mathcal{D} is nef,
- (2) $f^*(\mathcal{D}) = 0$, and
- (3) \mathcal{D} strictly positively intersects all the F-curves of type (1) – (4).

Proposition 7.10. *For $\ell \in \{1, 2, g-1, g\}$, there is a positive constant c_ℓ and nonnegative constant d_ℓ such that*

$$\mathbb{D}_{\ell,(1,\dots,1)}^{\mathfrak{sl}_2} = f^*(c_\ell D_{a,b}^\ell + d_\ell \mathcal{D}),$$

and $c_\ell D_{a,b}^\ell + d_\ell \mathcal{D}$ is a nef divisor on $\overline{M}_{2(g+1)}$. Here:

- (1) $c_1 D_{a,b}^1 = a\lambda - b\delta_0 - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \delta_{2i+1}$, where $c_1 = \frac{1}{4}$, $b \geq \frac{1}{2}$, and $a \geq 12b - 1$;
- (2) $c_2 D_{a,b}^2 = a\lambda - b\delta_0 - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \delta_{2i+1} - \sum_{i=1}^{\lfloor \frac{g+1}{2} \rfloor} \frac{4}{3} \delta_{2i}$, where $c_2 = \frac{4}{3}$, $b \geq \frac{8}{3}$, and $a \geq 12b - 1$;
- (3) $c_{g-1} D_{a,b}^{g-1} = a\lambda - b\delta_0 - \sum_{i=1}^g \frac{i(n-2i+1)}{n-1} \delta_i - \frac{3g+2}{n-1} \delta_{g+1}$,
where $c_{g-1} = \frac{1}{(g-1)}$, $b \geq \frac{1}{2} \max\{\frac{i(n-2i+1)}{n-1}\}_{i=1}^g$, and $a \geq 12b - 1$;
- (4) $c_g D_{a,b}^g = a\lambda - b\delta_0 - \sum_{i=1}^{g+1} \frac{i(n-2i+1)}{n-1} \delta_i$, where $c_g = 1$, $b \geq \frac{1}{2} \max\{\frac{i(n-2i+1)}{n-1}\}_{i=1}^{g+1}$ and $a \geq 12b - 1$.

We may take $d_1, d_2 \geq 0$. For $\ell \in \{g-1, g\}$, we may choose any d_ℓ such that $(c_\ell D_{a,b}^\ell + d_\ell \mathcal{D}) \cdot \mathcal{C} \geq 0$, where \mathcal{C} is any F curve of type (4).

Proof. By [15, Lemma 2.4], the pullback to $\overline{M}_{0,2(g+1)}$ of a divisor $D = a\lambda - \sum_{i=0}^{g+1} b_i \delta_i$ on $\overline{M}_{2(g+1)}$ along the flag map is

$$f^*D = \sum_{j=2}^{g+1} \left(\frac{j(n-j)}{(n-1)} b_1 - b_j \right) B_j, \quad \text{where} \quad B_j = \sum_{J \subset \{1, \dots, n\}, |J|=j} \delta_J,$$

where $n = 2(g+1)$.

Using this and the fact that $f^*\mathcal{D} = 0$, it is straightforward to check that for $\ell \in \{1, 2, g-1, g\}$, the divisors $c_\ell D_{ab}^\ell + d_\ell \mathcal{D}$ on $\overline{M}_{0,2(g+1)}$ pull back to $\mathbb{D}_{\ell, (1, \dots, 1)}^{\text{sl}_2}$. So it remains to check that for each $\ell \in \{1, 2, g-1, g\}$, the divisor $c_\ell D_{ab}^\ell + d_\ell \mathcal{D}$ is nef. Our main tool for proving that divisors on $\overline{M}_{0,2(g+1)}$ are nef will be to check the conditions of Theorem 7.8 and apply [16, Thm. 0.3].

First, we will analyze the cases $\ell = 1$ and $\ell = 2$.

It is easy to check that conditions (1) – (3) of Theorem 7.8 hold for D_{ab}^ℓ for all $\ell \in \{1, 2\}$ since we chose a and b to make this happen.

Condition (5) is just the combinatorial formulation that $(c_\ell D_{ab}^\ell) \cdot F_{i,j,k,\ell}^{2(g+1)} \geq 0$, where $F_{i,j,k,\ell}^{2(g+1)}$ on \overline{M}_n is the image of the F-curve $F_{i,j,k,\ell}$ on $\overline{M}_{0,n}$ under the flag map. In other words, this is equivalent to

$$f^*(c_\ell D_{ab}^\ell) \cdot F_{i,j,k,\ell} = \mathbb{D}_{\ell, (1, \dots, 1)}^{\text{sl}_2} \cdot F_{i,j,k,\ell} \geq 0,$$

which holds since $\mathbb{D}_{\ell, (1, \dots, 1)}^{\text{sl}_2}$ is nef.

This leaves condition (4).

We check condition (4) for D_{ab}^1 . Since b_k only depends on the parity of k , we need only consider two cases. If i and j have the same parity, then the equality reads $2 \geq 0$. If i and j have opposite parity, then the inequality reads $1 \geq 1$. So we may conclude that $D_{a,b}^1$ is nef. By [16, Thm. 4.7] \mathcal{D} is also nef, and hence for any nonnegative c_1, d_1 , the divisor $c_1 D_{a,b}^1 + d_1 \mathcal{D}$ is nef.

We next check condition (4) for D_{ab}^2 . Note that $b_i + b_j \geq 2$ while $b_{i+j} \leq \frac{4}{3}$, so (4) holds. Thus D_{ab}^2 is nef, and hence for any nonnegative c_2, d_2 , the divisor $c_2 D_{a,b}^2 + d_2 \mathcal{D}$ is nef.

Next, we will analyze the cases $\ell = g-1$ and $\ell = g$. The two divisors D_{ab}^{g-1} and D_{ab}^g are not nef by themselves, as condition (4) does not always hold. In particular, it is necessary to choose a sufficiently large d_{g-1} and d_g .

As before, the hypotheses on a, b, α , and β ensure that conditions (1) – (3) are satisfied. Condition (5) follows just as it did above since we know that $c_\ell D_{a,b}^\ell + d_\ell \mathcal{D}$ pulls back to a nef divisor on $\overline{M}_{0,n}$. This leaves condition (4). We know that \mathcal{D} has positive intersection with F-curves of type (4). Therefore, we simply need to choose d_ℓ sufficiently large that $(c_\ell D_{a,b}^\ell + d_\ell \mathcal{D}) \cdot \mathcal{C} \geq 0$ for all F-curves of type (4). There are only finitely many such curves to check (or, only finitely many inequalities of type $b_i + b_j \geq b_{i+j}$), so this can be arranged. Then $c_\ell D_{a,b}^\ell + d_\ell \mathcal{D}$ is F-nef, and hence nef by [16, Thm. 0.3]. □

Remark 7.11. *If given $\ell \in \{1, 2, g-1, g\}$, one could find a, b and d_ℓ such that $c_\ell D_{a,b}^\ell + d_\ell \mathcal{D}$ is semiample, then it would follow from Proposition 7.10 that the morphism given by $\mathbb{D}_{\ell, (1, \dots, 1)}^{\text{sl}_2}$ would factor through the flag locus. At the time of this writing, there are no divisors on $\overline{M}_{2(g+1)}$ that are known to be nef but not semiample.*

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